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ETS V: Phase Space Properties of Sequential Dynamical Systems

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Abstract

Sequential Dynamical Systems (SDS) [10] are a class of finite dynamical systems specifically designed to provide a framework for analysis and categorization of computer simulations. An SDS consists of (i) a finite undirected graph Y, where each vertex has associated a binary state, (ii) a collection of vertex labeled Y-local functions, and (iii) a permutation of the vertices of Y. The induced SDS is the map that results from applying the functions once to the states in the order given by the permutation. One theme of particular importance in SDS research is the study of its phase space, i.e. the finite unicyclic directed graph induced by the dynamical system [10].

In this paper we first establish what amounts to a morphism between two SDS having a specific relation, i.e. we assume that there is (i) a covering map between their corresponding base graphs $p : Y \to Z$, and (ii) their respective local functions and update schedules fulfill certain compatibility relations. This morphism then yields an embedding of the underlying phase spaces of the Z-SDS into the Y-SDS and therefore allows one to deduce phase space properties of the SDS over Y based on phase space properties of the SDS over Z. Second, we investigate the classes of graphs that can arise as images of covering maps for some fixed graph Y, as each of these graphs induces SDS that contain information about SDS over Y. In particular we will study this situation for the binary n-cube Q_2^n, where we show that there exists a covering map $p : Q_2^n \to K_{n+1}$ if and only if $2^n \equiv 0 \mod n+1$.

Key words: sequential dynamical systems, graph morphisms, covering maps, phase space embeddings, reduction, factorization.

1 Introduction

Computer simulations and networks of asynchronously updated nodes or vertices [1] , as for example gene-regulatory networks, catalytic networks or mobile communication networks, are typically difficult to analyze and categorize. In fact, there are not many conceptual frameworks available for a quantitative analysis of these systems, and most techniques do not seem to work well for asynchronous systems. Sequential dynamical systems (SDS) are a class of finite dynamical systems that provides a conceptual framework for asynchronous systems. In a straightforward way they encapsulate the three main features of simulations and asynchronous systems as they consist of a dependency graph Y, a collection of vertex labeled, Y-local functions [2] and a permutation of the functions according to which the update is performed. In the language of agent based simulations one may interpret a Y-local function as an agent, the graph Y represents the dependency backbone giving the actual communication links among agents, and finally the permutation functions as an update schedule of the system. SDS have been studied in [4, 10] and also in a slightly more generalized form in [6].

Let us discuss the three ingredients of an SDS in more detail: First, the underlying dependency graph is a finite, undirected graph, and each vertex has associated a binary state $x_i \in \mathbb{F}_2 = \{0, 1\}$. One may consider a vertex as an object, and an edge between vertices a and b indicates that the corresponding objects may exchange information about their states. Second, we have a collection or sequence of vertex labeled Boolean functions which perform the update of the states of the vertices. These functions are typically symmetric, i.e. they do not depend on the order of their arguments. Neither symmetry nor the choice of binary states represent severe conceptual restrictions for SDS. In particular, it is straightforward to generalize our results to SDS over arbitrary finite state spaces. Finally, the third component is the ordering or permutation of the vertices in the graph Y. This permutation gives the order in which the states are updated and has an oftentimes ignored impact on the structure and properties of the resulting system. This is to say that the scheduling of events is a crucial design feature. An SDS is thus a dynamical system of the

[1] the vertices are assumed to have "local" information

[2] A function is Y-local if it is defined over the states of a vertex i and the states of the Y-neighbors of i.

form $\phi : \mathbb{F}_2^n \to \mathbb{F}_2^n$. A generalization has been considered in [6] where certain vertices can be updated more than one time by the SDS.

Example: Consider the graph $Y = \mathrm{Circ}_4$ as shown in figure 1. To each vertex i

$$\begin{tikzcd} 1 \arrow[r, dash] \arrow[d, dash] & 2 \arrow[d, dash] \\ 4 \arrow[r, dash] & 3 \end{tikzcd}$$

Fig. 1. The circle graph on 4 vertices, Circ_4.

in Y we associate a state $x_i \in \{0,1\} = \mathbb{F}_2$. The parity function $\mathrm{par}_3 : \mathbb{F}_2^3 \to \mathbb{F}_2$ is defined by $\mathrm{par}_3(x_1, x_2, x_3) = x_1 + x_2 + x_3$. In the usual enumeration scheme of elementary CA rules this is the rule labeled 150. We introduce the functions $\mathrm{Par}_i : \mathbb{F}_2^4 \to \mathbb{F}_2^4$, $1 \le i \le 4$ by

$$\begin{aligned}
\mathrm{Par}_1(x_1, x_2, x_3, x_4) &= (\mathrm{par}_3(x_1, x_2, x_4), x_2, x_3, x_4), \\
\mathrm{Par}_2(x_1, x_2, x_3, x_4) &= (x_1, \mathrm{par}_3(x_1, x_2, x_3), x_3, x_4), \\
\mathrm{Par}_3(x_1, x_2, x_3, x_4) &= (x_1, x_2, \mathrm{par}_3(x_2, x_3, x_4), x_4), \\
\mathrm{Par}_4(x_1, x_2, x_3, x_4) &= (x_1, x_2, x_3, \mathrm{par}_3(x_1, x_3, x_4)).
\end{aligned}$$

Thus the map Par_i updates the state of vertex i based on the states of i and its neighbors in Y and leaves all other states fixed. We apply these maps to the state $x = (1,1,0,0)$ in the order $\pi = (1,2,3,4)$. At each stage we use the value of the previous function as the input to the next function, i.e.

$$(1,1,0,0) \overset{\mathrm{Par}_1}{\mapsto} (0,1,0,0) \overset{\mathrm{Par}_2}{\mapsto} (0,1,0,0) \overset{\mathrm{Par}_3}{\mapsto} (0,1,1,0) \overset{\mathrm{Par}_4}{\mapsto} (0,1,1,1) \tag{1}$$

Thus we have $\mathrm{Par}_4 \circ \mathrm{Par}_3 \circ \mathrm{Par}_2 \circ \mathrm{Par}_1(1,1,0,0) = (0,1,1,1)$. The composition of maps $\mathrm{Par}_4 \circ \mathrm{Par}_3 \circ \mathrm{Par}_2 \circ \mathrm{Par}_1$ is a sequential dynamical system (SDS). Specifically, it is the SDS over the graph Circ_4 induced by the parity function $\mathrm{par}_3 : \mathbb{F}_2^3 \to \mathbb{F}_2$ with ordering $\pi = (1,2,3,4)$. We denote this by $[\mathbf{Par}_{\mathrm{Circ}_4}, \pi]$. Clearly, a different update order may give a different result. By iterating the map $\phi = [\mathbf{Par}_{\mathrm{Circ}_4}, \pi]$ we obtain the *orbit* of $(1,1,0,0)$, i.e.

$$\begin{tikzcd} (1,1,0,0) \arrow[rr, "\phi"] & & (0,1,1,1) \arrow[dl, "\phi"] \\ & (0,0,0,1). \arrow[ul, "\phi"] & \end{tikzcd}$$

The *phase space* of the SDS ϕ is the union of all such cycles and possible transients.

One central question in SDS analysis is how rescheduling influences the dynamical system. In particular, this amounts to a design question for computer simulations since the actual implementation dictates a corresponding choice of schedule. The question is to what extend would a different implementation produce a "different" system?

Example: Let ϕ_1 and ϕ_2 be the two SDS induced by the parity function over the graph Circ_5 (figure 2) with the schedules $\pi_1 = (1, 2, 3, 4, 5)$ and $\pi_2 = (1, 3, 5, 2, 4)$, respectively.

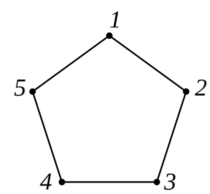

Fig. 2. The circle graph on 5 vertices, Circ_5.

It is shown in [2] that SDS induced by the parity function are invertible as maps independent of the underlying base graph. Accordingly, the phase spaces of ϕ_1 and ϕ_2 consist of cycles (periodic points) and isolated points (fixed points). We will show that the schedules π_1 and π_2 induce SDS with significant differences in both local and global dynamics. Straightforward calculations show that ϕ_1 has two fixed points, one 2-cycle, three 4-cycles and two 8-cycles. On the other hand ϕ_2 exhibits two fixed points, one 2-cycle, two 3-cycles, one 4 cycle, one 6-cycle and one 12-cycle. The entire phase spaces are presented in figures 3 and 4 respectively.

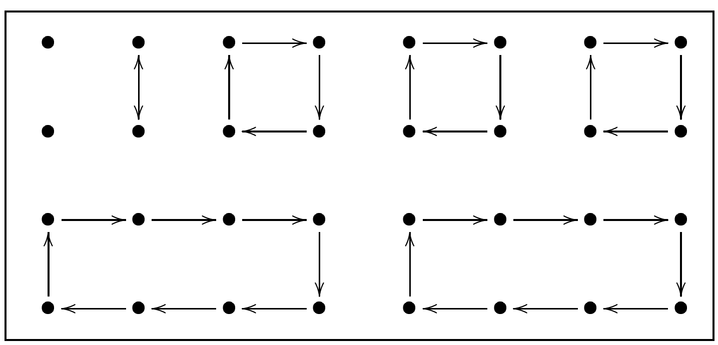

Fig. 3. The phase space of the SDS ϕ_1.

Remarkably, the structural difference in terms of periodic orbits is not the only difference between these two SDS. In fact, ϕ_2, in contrast to ϕ_1 has a

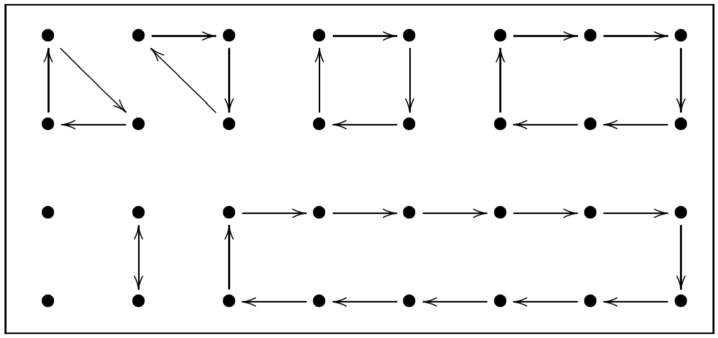

Fig. 4. The phase space of the SDS ϕ_2.

dense[3] *periodic orbit.* The 12-cycle for ϕ_2 which is dense in $\mathbb{F}_2^n$ is shown in figure 5.

$$\begin{tikzcd}
(1,0,0,0,0) \arrow[r] & (1,1,0,1,1) \arrow[r] & (1,0,0,0,1) \arrow[r] & (0,0,0,1,1) \arrow[r] & (1,0,1,1,1) \arrow[d] \\
(1,1,1,0,1) \arrow[u] & & & & (0,0,0,1,0) \arrow[d] \\
(0,1,0,0,0) \arrow[u] & (1,1,1,0,0) \arrow[l] & (0,1,1,1,0) \arrow[l] & (0,0,1,0,0) \arrow[l] & (0,1,1,1,1) \arrow[l]
\end{tikzcd}$$

Fig. 5. The dense 12-cycle in the phase space of the SDS ϕ_2.

Obviously, the analysis of the dynamics of a simulation or network now becomes a question about the structure of the phase space of its underlying SDS. One important research theme for SDS is deducing properties of the phase space based on known quantities, typically Y-local information such as, e.g. the graph Y, the local functions and the schedule. For various results on, e.g. reversibility/invertibility and fixed points we refer to [2–4, 10, 11]. Typically, the phase space of an SDS has more than one attractor or component, and consequently a time series will only visit parts of phase space. Thus, in the context of computer simulations, there will be valid states or regimes that are never realized. Accordingly, one is interested in constructing a "reduced" simulation system capable of producing a somewhat related dynamics in the "essential" regimes and that ideally disposes of the "non-essential" regimes. In this paper we address this question by establishing an embedding of SDS phase spaces under certain conditions. Explicitly, we will show how to relate an SDS ϕ over a graph Y and an SDS ψ over a smaller sized graph Z if there exists a covering map $p : Y \to Z$. This idea can be illustrated as follows:

[3] A periodic orbit $P = \{p_1, \ldots, p_k\}$ is dense in $\mathbb{F}_2^n$ if the set $\{x \in \mathbb{F}_2^n \mid d_H(x, P) \leq 1\}$ equals $\mathbb{F}_2^n$, where d_H denotes the usual Hamming metric.

Example: Let $Y = Q_2^3$ and $Z = K_4$, see figure 6. We will consider sequential

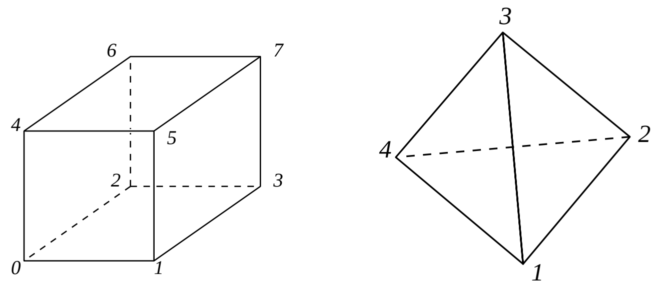

Fig. 6. The graph K_4 is a covering image of the graph Q_2^3.

dynamical systems over Q_2^3 and K_4 induced by the parity function. We first observe that there exists a covering map $p : Q_2^3 \to K_4$ given by $p^{-1}(\{1\}) = \{0,7\}$, $p^{-1}(\{2\}) = \{1,6\}$, $p^{-1}(\{3\}) = \{2,5\}$, and $p^{-1}(\{4\}) = \{3,4\}$. Now, p naturally induces an embedding $\tau : \mathbb{F}_2^4 \to \mathbb{F}_2^8$ by $(\tau(x))_k = x_{p(k)}$, that is,

$$\tau(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4, x_4, x_3, x_2, x_1) \ .$$

Let $\pi = (1,2,3,4) \in S_4$, and let $\pi_p = (0,7,1,6,2,5,3,4)$. We now have a commutative diagram

$$\begin{tikzcd}
\mathbb{F}_2^4 \arrow[r, "{[\mathrm{Par}_{K_4},(1,2,3,4)]}"] \arrow[d, "\tau"] & \mathbb{F}_2^4 \arrow[d, "\tau"] \\
\mathbb{F}_2^8 \arrow[r, "{[\mathrm{Par}_{Q_2^3},\pi_p]}"] & \mathbb{F}_2^8 \ .
\end{tikzcd}$$

Let further $x = (1,0,0,0)$. We have $[\mathrm{Par}_{K_4}, (1,2,3,4)](1,0,0,0) = (1,1,0,0)$, and further iterations give the orbit

$$\begin{tikzcd}
(1,0,0,0) \arrow[r] & (1,1,0,0) \arrow[r] & (0,1,1,0) \arrow[dl] \\
(0,0,0,1) \arrow[u] & (0,0,1,1) \arrow[l] \ . &
\end{tikzcd}$$

Note that

$$\begin{tikzcd}
(1,0,0,0) \arrow[r, "{[\mathrm{Par}_{K_4},(1,2,3,4)]}"] \arrow[d, "\tau"] & (1,1,0,0) \arrow[d, "\tau"] \\
(1,0,0,0,0,0,0,1) \arrow[r, "{[\mathrm{Par}_{Q_2^3},\pi_p]}"] & (1,1,0,0,0,0,1,1) \ .
\end{tikzcd}$$

By applying the map τ to the cycle of $(0,0,0,0)$ under $[\mathrm{Par}_{K_4}, (1,2,3,4)]$ it is easily verified that we obtain the orbit of $(1,0,0,0,0,0,0,1)$ under $[\mathrm{Par}_{Q_2^3}, \pi_p]$:

$$\begin{tikzcd}
(1,0,0,0,0,0,0,1) \arrow[r] & (1,1,0,0,0,0,1,1) \arrow[r] & (0,1,1,0,0,1,1,0) \arrow[dl] \\
(0,0,0,1,1,0,0,0) \arrow[u] & (0,0,1,1,1,1,0,0) \arrow[l] &
\end{tikzcd}\ .$$

A more detailed calculation shows that the entire phase space of the SDS $[\mathrm{Par}_{K_4}, (1,2,3,4)]$ can be embedded[4] in the phase space of $[\mathrm{Par}_{Q_2^3}, \pi_p]$.

The above example is a particular instance of theorem 4. In general, two SDS are "dynamically" related by an embedding of their phase spaces if (i) the local functions are identical, (ii) there exists a covering map between their underlying base graphs, and (iii) their schedules fulfill a specific compatibility relation. Under these conditions the smaller system could potentially exhibit "key" properties of the larger system. Clearly, there can be several covering maps $p_i : Y \to Z_i$ for a fixed graph Y and varying graphs Z_i. These maps can be viewed as some kind of "factorization" of the SDS over Y into factors which are SDS over the respective graphs Z_i.

The paper is organized as follows: in section 2 we provide basic definitions and terminology for SDS. In section 3 we give the formulation of the main theorem and finally in section 4 we investigate how to factorize SDS over n-cubes.

2 Basic terminology and definitions

Let Y be a labeled graph with vertex-set $\mathrm{v}[Y] = \mathbb{N}_n = \{1, 2, 3, \ldots, n\}$, which we write as $Y < K_n$. The edge-set of Y is denoted by $\mathrm{e}[Y]$. A morphism between graphs Y and Y' is a pair $\phi = (\phi_1, \phi_2)$ with $\phi_1 : \mathrm{v}[Y] \to \mathrm{v}[Y']$ and $\phi : \mathrm{e}[Y] \to \mathrm{e}[Y']$ such that

$$\forall e = \{i, j\} \in \mathrm{e}[Y] : \quad \phi_2(e) = \{\phi_1(i), \phi_1(j)\}.$$

Thus, adjacent vertices in Y are mapped to i) adjacent vertices in Y' or ii) to the same vertex in Y'. A morphism of directed graphs also preserves the direction of edges. A graph morphism $\phi : Y \to Y'$ is *locally bijective*

[4] The former SDS has one fixed point and three orbits of length 5 while the latter SDS has one fixed point and 51 orbits of length 5.

(surjective) if

$$\forall i \in \mathrm{v}[Y] : \phi|_{B_{1,Y}(i)} : B_{1,Y}(i) \to B_{1,Y'}(\phi(i))$$

is bijective (surjective). In the following we will use the term *covering map* instead of locally bijective graph morphism. Note that a covering map (locally bijective graph morphism) does not have to be bijective, see for example figure 6.

Let $S_{1,Y}(i)$ be the set of Y-vertices that are adjacent to vertex i, let $\delta_i = |S_{1,Y}(i)|$ and let $d = \max_{i \in \mathbb{N}_n} \delta_i$. The increasing sequence of elements of $S_{1,Y}(i)$ preceded by i is denoted by

$$\tilde{B}_{1,Y}(i) = (i, j_1, \ldots, j_{\delta_i}). \tag{2}$$

A function $f : E^k \to F$, where E and F are vector spaces, is quasi-symmetric if for all $x \in E^k$ and all permutations $\sigma \in 1 \times S_{k-1}$ we have $f(\sigma \cdot x) = f(x)$ where $\sigma \cdot x$ is the permutation action on k-tuples given by $\sigma \cdot x = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$. We write $\mathrm{QSymm}(E^n, F)$ for the set of all quasi-symmetric functions from E^n to F.

To each vertex i of Y we associate a state $x_i \in \mathbb{F}_2$, and we write $x = (x_1, x_2, \ldots, x_n)$ for the system state. For each $k = 1, \ldots, d+1$ we have a function $f_k \in \mathrm{QSymm}(\mathbb{F}_2^k, \mathbb{F}_2)$, and for each vertex i we introduce a map

$$\mathrm{proj}_Y[i] : \mathbb{F}_2^n \to \mathbb{F}_2^{\delta_i+1}, \quad (x_1, \ldots, x_n) \mapsto (x_i, x_{j_1}, \ldots, x_{j_{\delta_i}}). \tag{3}$$

The map projects from the full n-tuple x down to the states vertex i needs for updating its state. For each $i \in \mathbb{N}_n$ there is a Y-local map $F_{i,Y} : \mathbb{F}_2^n \to \mathbb{F}_2^n$ given by

$$\begin{aligned} y_i &= f_{\delta_i+1} \circ \mathrm{proj}_Y[i], \\ F_{i,Y}(x) &= (x_1, \ldots, x_{i-1}, y_i(x), x_{i+1}, \ldots, x_n), \end{aligned} \tag{4}$$

as in the example on page 3. The function $F_{i,Y}$ updates the state of vertex i and leaves all other states fixed. We refer to the sequence $(F_{i,Y})_i$ as F_Y. Note that for each graph $Y < K_n$ a sequence $(f_k)_{1 \le k \le n}$ induces a sequence F_Y, i.e. we have a map $\{Y < K_n\} \to \{F_Y\}$. We define the map $[F_Y, \ ] : S_n \to \mathrm{Map}(\mathbb{F}_2^n, \mathbb{F}_2^n)$ by

$$[F_Y, \pi] = \prod_{i=1}^{n} F_{\pi(i),Y}, \tag{5}$$

where product denotes ordinary function composition.

Definition 1 (Sequential Dynamical System) *Let $Y < K_n$, let $(f_k)_k$ for $1 \leq k \leq d(Y)+1$ be a sequence of quasi-symmetric functions as above, and let $\pi \in S_n$ The sequential dynamical system (*SDS*) over Y induced by $(f_k)_k$ with respect to the ordering π is $[F_Y, \pi]$.*

Definition 2 *The digraph $\Gamma[F_Y, \pi]$ associated to the* SDS *$[F_Y, \pi]$ is the directed graph having vertex-set $\mathbb{F}_2^n$ and directed edges $\{(x, [F_Y, \pi](x)) \mid x \in \mathbb{F}_2^n\}$.*

For some examples of phase spaces refer to figure 3 and figure 4.

3 Factorization of SDS

The idea behind the factorization of a given SDS is to relate it to SDS defined over simpler graphs. Accordingly, the term "relation" has to be made precise which amounts to defining what a morphism between SDS is:

Definition 3 *Let $[F_Z, \sigma]$ and $[F_Y, \pi]$ be two* SDS*. An* SDS*-morphism between $[F_Z, \sigma]$ and $[F_Y, \pi]$ is a pair (ϕ, Φ) where $\phi : Y \to Z$ is a graph morphism and where $\Phi : \Gamma[F_Z, \sigma] \to \Gamma[F_Y, \pi]$ is digraph morphism.*

Given a graph morphism $\phi : Y \to Z$ we want to relate the dynamics of SDS over the two graphs Y and Z. The local functions will be the same for the two graphs unless otherwise stated. To begin, we relate update schedules for Y and Z via ϕ. Assume $|\,\mathrm{v}[Y]| = n$ and $|\,\mathrm{v}[Z]| = m$ and let $\phi^{-1}(i) = \{i_1, \dots, i_{l_i}\}$ where $i_1 < \dots i_{l_i}$ for $1 \leq i \leq m$. Define the map $\eta_\phi : S_m \to S_n$ by

$$\eta_\phi(\pi = (\pi_1, \pi_2, \dots, \pi_m)) = (\pi_{11}, \dots, \pi_{1 l_{\pi_1}}, \dots, \pi_{m1}, \dots, \pi_{1 l_{\pi_m}}). \qquad (6)$$

For instance, in the example with $\phi : Q_2^3 \to K_4$, we have $\eta_\phi(4,3,2,1) = (3,4,2,5,1,6,0,7)$.

Similarly, we define the map $\tau : \mathbb{F}_2^m \to \mathbb{F}_2^n$ by

$$(\tau(x))_k = x_{\phi(k)}. \qquad (7)$$

The dynamics of SDS over Y and Z can now be related in the following way [12]:

Theorem 4 *Let Y and Z be loop-free connected graphs, let $\phi : Y \to Z$ be a covering map, and let $(f_i)_i$ be a fixed sequence of Boolean quasi-symmetric functions. Then the map τ induces a natural embedding*

$$T : \Gamma[F_Z, \pi] \hookrightarrow \Gamma[F_Y, \eta_\phi(\pi)]. \tag{8}$$

Example: Let $\sigma = (0, 7, 1, 6, 2, 5, 3, 4)$. To illustrate Theorem 4 we show how to relate the phase space of the two SDS $[\mathbf{Min}_{K_4}, \mathrm{id}_4]$ and $[\mathbf{Min}_{Q_2^3}, \sigma]$. We already discussed the covering $\phi : Q_2^3 \longrightarrow K_4$ and observe that $\eta_\phi(\mathrm{id}_4) = \sigma$. In [1] we have shown that $[\mathbf{Min}_{K_4}, \mathrm{id}_4]$ has exactly two 5-cycles and no fixed points. The two 5-cycles are shown in the top row of figure 7. For convenience we use the map

$$\xi_i : \mathbb{F}_2^i \longrightarrow \mathbb{N}, \qquad \xi_i(x_1, \ldots, x_i) = \sum_{j=0}^{i} x_j \cdot 2^{j-1}$$

to encode states (binary tuples), and we have, e.g. $(1, 1, 0, 1) \mapsto 1+2+8 = 11$. It is straightforward to see that the phase space of $[\mathbf{Min}_{K_4}, \mathrm{id}_4]$ is indeed embedded in the phase space of $[\mathbf{Min}_{Q_2^3}, \sigma]$.

We remark that $[\mathbf{Min}_{Q_2^3}, \eta_\phi(\mathrm{id}_4)]$ has two fixed points in addition to the two 5-cycles shown in the last row in figure 7. These fixed points are related by the graph automorphism $\gamma = (07)(16)(25)(34)$, and consequently, so are their transients. Stated differently, the two components in $\Gamma[\mathbf{Min}_{Q_2^3}, \eta_\phi(\mathrm{id}_4)]$ containing the fixed points are isomorphic. We present their structure in detail in figure 8.

It will usually be more feasible to analyze the SDS over the smaller graph, and a particularly interesting case is the following:

Proposition 5 *Let $\pi \in S_{n+1}$ and assume $2^n \equiv 0 \bmod n+1$. Then the* SDS $[\mathbf{Par}_{Q_2^n}, \eta_\phi(\pi)]$ *has a periodic orbit of length $n + 2$.*

PROOF. We will apply Theorem 4 in order to show that the phase space of $\Phi = [\mathbf{Par}_{K_n}, \pi]$ can be embedded into the phase space of $\Psi = [\mathbf{Par}_{Q_2^n}, \eta_\phi(\pi)]$. Then we prove the existence of periodic orbit of length $n+2$ for $\Phi = [\mathbf{Par}_{K_n}, \pi]$ and the proposition follows.

In order to apply Theorem 4 we need the existence of a covering map $\phi : Q_2^n \to K_{n+1}$ for $2^n \equiv 0 \bmod n+1$ which is not entirely trivial and is proved

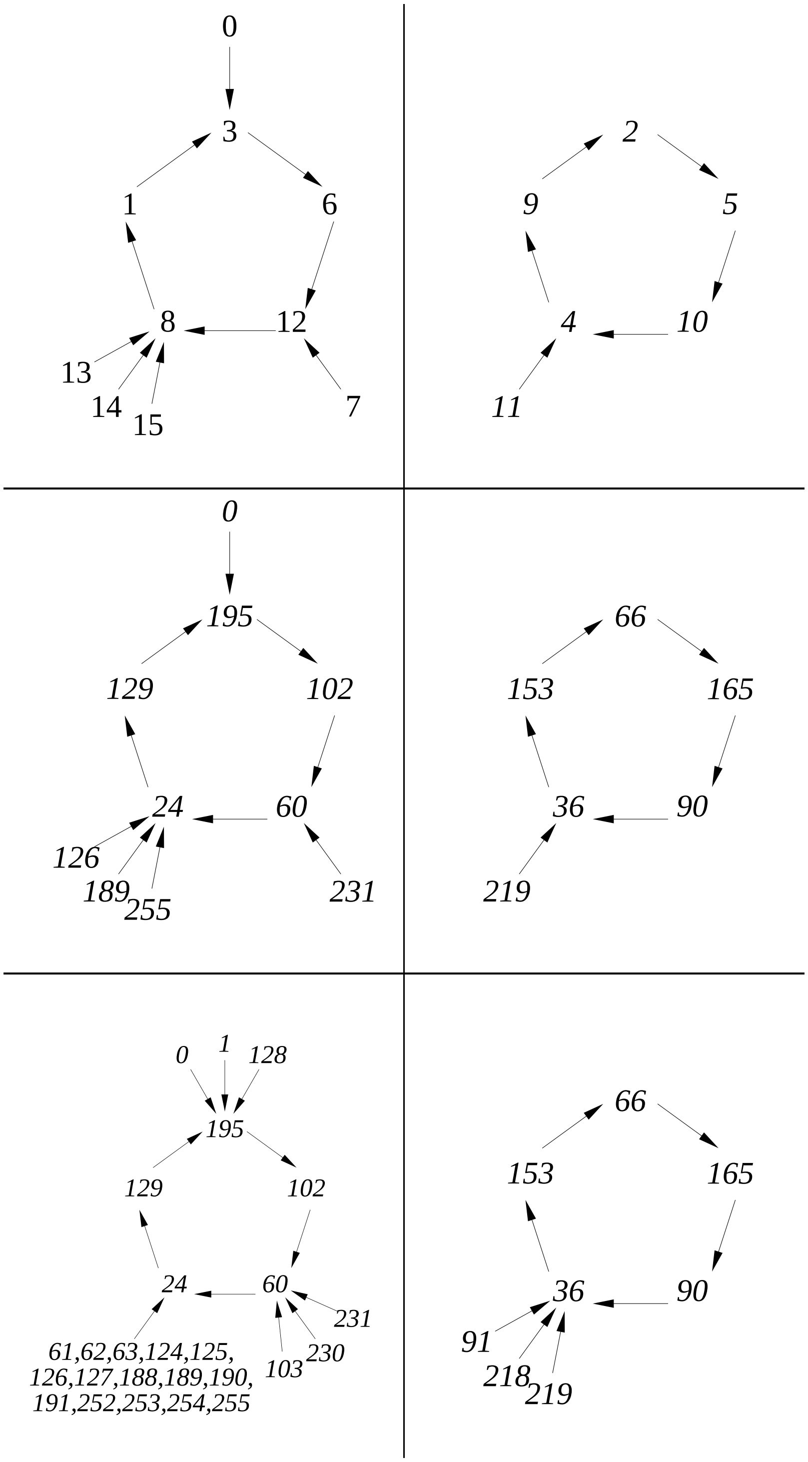

Fig. 7. The top row shows the two five-cycles in $[\mathbf{Min}_{K_4}, \mathrm{id}]$. The second row shows the images of the top cycles under τ_ϕ, and the last row shows the corresponding periodic cycles in the digraph $\Gamma[\mathbf{Min}_{Q_2^3}, \eta_\phi(\mathrm{id}_4)]$.

in section 4.

Accordingly, it remains to show that Φ exhibits a periodic orbit of length $n+2$.

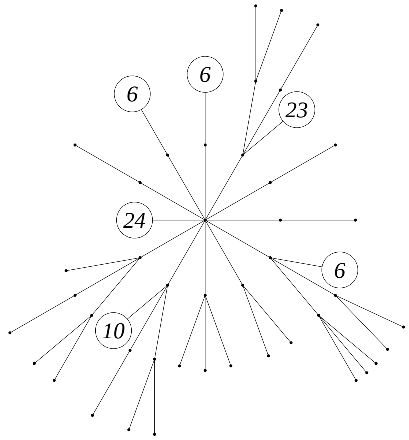

Fig. 8. The structure of the components in $\Gamma[\mathbf{Min}_{Q_2^3}, \eta_\phi(\mathrm{id}_4)]$ containing a fixed point. A single filled circle depicts a single state, while a circled number i depicts that there are i direct predecessors that do not have any predecessors themselves.

Without loss of generality we may choose $\pi = \mathrm{id}_{n+1}$ since all other schedules induce isomorphic SDS [10]. We next observe that $\mathrm{par}_n : \mathbb{F}_2^n \to \mathbb{F}_2$ satisfies the following functional relation:

$$\psi(x_1, \ldots, x_{n-1}, \psi(x_1, \ldots, x_n)) = x_n. \tag{9}$$

As a consequence of this we derive

$$\begin{aligned} x = (x_1, x_2, \ldots, x_n) &\overset{1}{\mapsto} (\mathrm{par}_n(x), x_2, x_3, \ldots, x_n) \\ &\overset{2}{\mapsto} (\mathrm{par}_n(x), \mathrm{par}_n(\mathrm{par}_n(x), x_2, \ldots, x_n), x_3, \ldots, x_n) \\ &= (\mathrm{par}_n(x), x_1, x_3, \ldots, x_n) \\ &\vdots \\ &\overset{n}{\mapsto} (\mathrm{par}_n(x), x_1, x_2, \ldots, x_{n-1}), \end{aligned}$$

where $\overset{i}{\mapsto}$ denotes the update of state x_i. Therefore we obtain the commutative diagram

$$\begin{tikzcd} \mathbb{F}_2^n \arrow[r, "{[\mathrm{Par}_{K_n}, \mathrm{id}]}"] \arrow[d, "\iota_{\mathrm{par}_n}"'] & \mathbb{F}_2^n \\ \hat{\mathbb{F}}_2^n \arrow[r, "\sigma_{n+1}"] & \hat{\mathbb{F}}_2^n, \arrow[u, "\mathrm{proj}"'] \end{tikzcd} \tag{10}$$

where

$$\mathrm{proj}(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n), \tag{11}$$

$$\iota_{\mathrm{par}_n}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \mathrm{par}_n(x_1, \ldots, x_n)), \tag{12}$$

$$\sigma_{n+1}(x_1, x_2, \ldots, x_{n+1}) = (x_{n+1}, x_1, \ldots, x_n), \tag{13}$$

and $\hat{\mathbb{F}}_2^n = \{x \in \mathbb{F}_2^{n+1} \mid x_{n+1} = \mathrm{par}_n(x_1, \ldots, x_n)\}$.
Note that $\mathrm{proj} : \hat{\mathbb{F}}_2^n \to \mathbb{F}_2^n$ and $\iota_{\mathrm{par}_n} : \mathbb{F}_2^n \to \hat{\mathbb{F}}_2^n$ are inverse with respect to each other. Similarly we obtain $[\mathrm{Par}_{K_n}, \mathrm{id}]^{(2)}(x) = (x_n, \mathrm{par}_n(x), x_1, x_2, \ldots, x_{n-2})$ and in general $[\mathrm{Par}_{K_n}, \mathrm{id}]^{(k)} = \mathrm{proj} \circ \sigma_{n+1}^k \circ \iota_{\mathrm{par}_n}$, whence the order of an orbit of $[\mathrm{Par}_{K_n}, \mathrm{id}]$ is a divisor of $n+1$. Furthermore, it is easy to see that the orbit containing the state $(1, 0, 0, \ldots, 0)$ always have length $n+2$. Explicitly we have for $n = 7$

$$\begin{tikzcd}
(1000000) \arrow[r] & (1100000) \arrow[r] & (0110000) \arrow[r] & (0011000) \arrow[d] \\
(0000001) \arrow[u] & (0000011) \arrow[l] & (0000110) \arrow[l] & (0001100) \arrow[l]
\end{tikzcd} .$$

Thus we can deduce from Theorem 4 that $[\mathbf{Par}_{Q_2^n}, \eta_\phi(\pi)]$ has a periodic orbit of length $n+2$ and the proof of the proposition is complete.

The above proposition raises the following question: Given a graph Y, what are the covering maps ψ over Y, and what are the covering images Z, $\psi : Y \to Z$? This question is investigated in some detail in the next section for n-cubes.

4 Covering maps over the n-cube

Recall that a covering map $p : Y \to Z$ is the same as a locally bijective graph morphism from Y to Z. The covering map $\phi : Q_2^3 \to K_4$ turns out to be a special instance of a class of covering maps over n-cubes. The key idea here is to consider Q_2^3 as a Cayley graph and K_4 as an orbit space with respect to a regularly acting subgroup of Q_2^3-automorphisms.

Covering maps have been studied in [5] as follows: let $S\Gamma$ denote the set of arcs or sides of the graph Γ. Thus each edge $\{u, v\}$ gives rise to two sides, (u, v) and (v, u). Let G be any group. A G-chain on Γ is a map $\phi : S\Gamma \to G$ such that $\phi(u, v) = (\phi(v, u))^{-1}$ for all sides (u, v) of Γ. The *covering map* $\hat{\Gamma} = \hat{\Gamma}(G, \phi)$ of Γ with respect to a given G-chain ϕ on Γ is the graph with $\mathrm{v}[\hat{\Gamma}] = G \times \mathrm{v}[\Gamma]$ and where vertices (g_1, v_1) and (g_2, v_2) are joined by an edge iff $(v_1, v_2) \in S\Gamma$ and $g_2 = g_1\phi(v_1, v_2)$. $\hat{\Gamma}$ is clearly well-defined. Note that the 3-cube is isomorphic to the covering graph $\hat{K_4}(\mathbb{F}_2, \phi)$ of K_4 where ϕ is the $\mathbb{F}_2$-chain assigning 1 to each side of K_4. Moreover we note that the graph G_5 on the third row in figure 10 is isomorphic to the covering graph $\hat{K}_8(\mathbb{F}_2,\phi)$, where ϕ again is the $\mathbb{F}_2$-chain assigning 1 to each side of K_8. A similar approach for constructing covering maps uses so-called "voltage graph" [7–9]. Voltage graphs and G-chains are closely related.

In this section we will develop our above orbit-space idea in some generality and obtain new covering maps over n-cubes. We will first show how the existence of subgroups $H < \mathbb{F}_2^n$ with certain properties can be used to obtain covering maps. Specifically, a subgroup H of the n-cube Q_2^n will, under certain conditions, induce a covering $\phi : Q_2^n \to H \setminus Q_2^n$ where $H \setminus Q_2^n$ has vertex set Q_2^n/H and two vertices $\bar{u}$, $\bar{v}$ are adjacent iff there are elements $u \in \bar{u}$ and $v \in \bar{v}$ such that $\{u,v\} \in \mathrm{e}[Q_2^n]$.

Lemma 6 *For any subgroup $H' < \mathbb{F}_2^n$ with $[\mathbb{F}_2^n : H'] \geq n+1$ there exists an H that has the property $H(x) \cap H(y) = \varnothing$ for*

$y \in \{0, e_1, \dots, e_n\}$.

. [sketch] Let G' be a group such that $[\mathbb{F}_2^n : G'] \geq n+1$ holds. The show that then there exists a set of representatives Φ of $G' < \mathbb{F}_2^n$ that an $\mathbb{F}_2^n$-basis. Then one considers the $\mathbb{F}_2^n$-homomorphism f defined by $_i$, for $i = 1, \dots, n$. Clearly, the subgroup $G = f(G')$ has the property

$$\mathbb{F}_2^n = G(0) \cup \bigcup_{i=1}^{n} G(e_i) \cup \bigcup_{j=s+1}^{[\mathbb{F}_2^n : G]-1-(n-s)} G(f(w_j)) \ ,$$

ne lemma.

ion 7 *For each subgroup $H < \mathbb{F}_2^n$ with the property $H(x) \cap H(y) = \varnothing$; $x, y \in \{0, e_1, \dots, e_n\}$ the graph $H \setminus Q_2^n$ is connected, undirected and and the natural projection*

$$\pi_H : Q_2^n \longrightarrow H \setminus Q_2^n, \quad v \mapsto H(i)$$

ing-map.

. We have to show that the π_H-induced restriction mapping

$$\mathrm{res}_{\mathrm{Star}_{Q_2^n}(\xi)}(\pi_H) : \mathrm{Star}_{Q_2^n}(\xi) \longrightarrow \mathrm{Star}_{H\setminus Q_2^n}(\pi_H(\xi)) \tag{14}$$

is an isomorphism for arbitrary $\xi \in \mathbb{F}_2^n$. By construction, $\mathrm{res}_{\mathrm{Star}_{Q_2^n}(\xi)}(\pi_H)$ is surjective and $H(x) \cap H(y) = \varnothing$ for $x \neq y$; $x, y \in \{0, e_1, \ldots, e_n\}$ is equivalent to $H(x+\xi) \cap H(y+\xi) = \varnothing$ for $x \neq y$; $x, y \in \{0, e_1, \ldots, e_n\}$ for any $\xi \in \mathbb{F}_2^n$. Therefore $\mathrm{res}_{\mathrm{Star}_{Q_2^n}(\xi)}(\pi_H)$ is injective and the proof of the proposition is complete.

Example: [Covering maps over Q_2^4 and Q_2^7]

We have constructed all covering images of the form $H \setminus Q_2^n$ for $n = 4$ and $n = 7$. There are two non-isomorphic covering images when $n = 4$ and five non-isomorphic covering images when $n = 7$. Moreover, for $n = 4$ explicit computations show that the two covering images of the form $H \setminus Q_2^4$ are the only covering images.

The covering images for $n = 4$ and $n = 7$ are shown in figure 9 and figure 10, respectively.

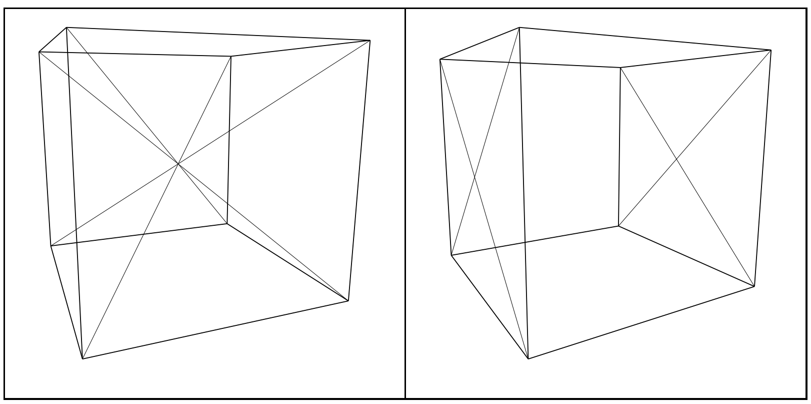

Fig. 9. The only non-isomorphic graphs of the form $H \setminus Q_2^4$.

Let $H_1, H_2 < Q_2^n$ be subgroups satisfying the condition of Lemma 6, that is, $H_i(x) \cap H_i(y) = \varnothing$ for $x \neq y$; $x, y \in \{0, e_1, \ldots, e_n\}$. Assume there exists a graph isomorphism $\bar{\psi} : H_1 \setminus Q_2^n \to H_2 \setminus Q_2^n$. Can $\bar{\psi}$ be lifted to a graph isomorphism $\psi : Q_2^n \to Q_2^n$? Equivalently, does there exist a graph isomorphism ψ such that the following diagram is commutative:

$$\begin{tikzcd} Q_2^n \arrow[r, "\psi"] \arrow[d, "p_1"'] & Q_2^n \arrow[d, "p_2"] \\ H_1 \setminus Q_2^n \arrow[r, "\bar{\psi}"] & H_2 \setminus Q_2^n \end{tikzcd} \tag{15}$$

Here $p_i : Q_2^n \to H_i \setminus Q_2^n$ is the standard coset projection map. This question is

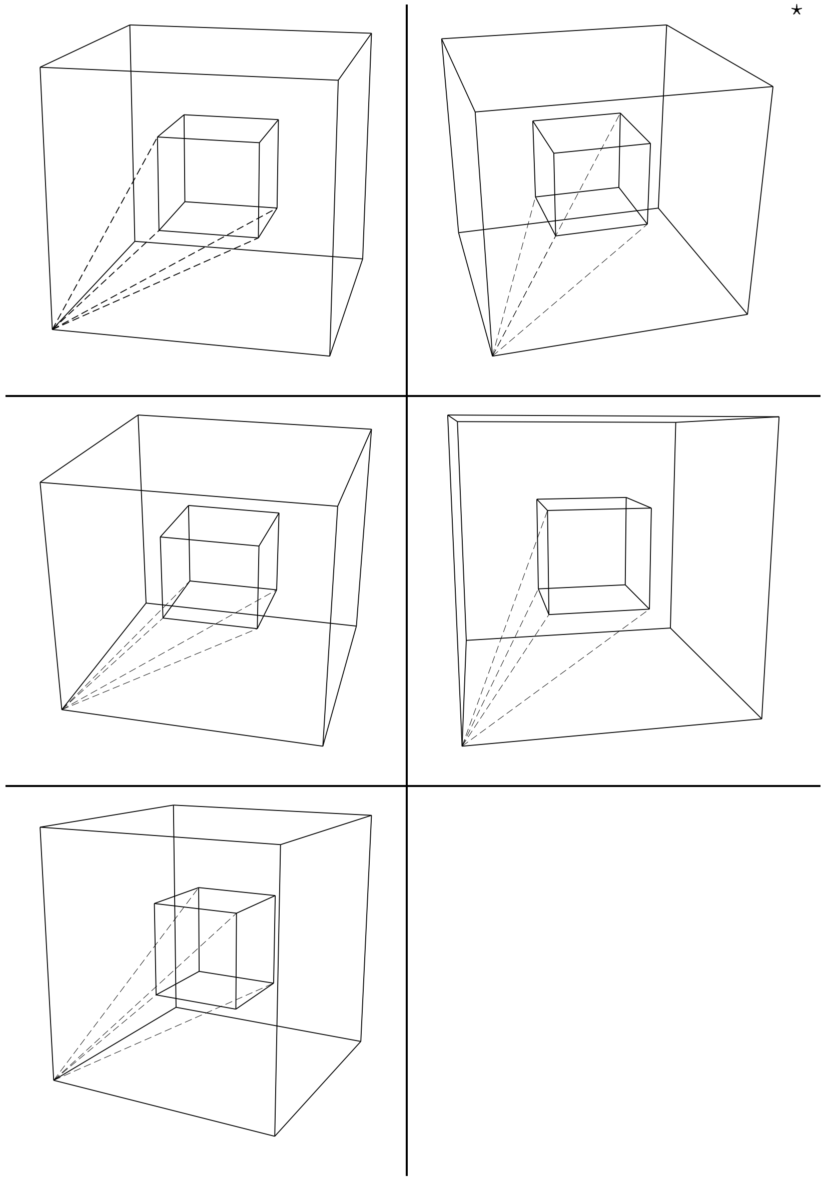

Fig. 10. All non-isomorphic graphs of the form $H \setminus Q_2^7$ on 16 vertices. In each case there are two interlocked cubes. The dotted edges show how the two cubes are connected. The other vertices of the outer cube are similarly connected, but the edges are not shown to ease visualization. Note that there are 4 lines in figure on the top right $\star$. The two lines connecting the outer cube to the extreme points of the inner cube coincide.

of particular interest since it reflects on the relation between the automorphism group of Q_2^n and its corresponding orbit-space.

Let $\psi(0) = 0$. Then we can compute the values of ψ on its neighbors and raise the question whether or not ψ can be lifted. The following example shows that in general this is impossible: Let $H < Q_2^7$ be the subgroup

$$\begin{aligned} H = \{&(0000000), (1101000), (1010100), (0111100), \\ &(0110010), (1011010), (1100110), (0001110), \\ &(1110001), (0011001), (0100101), (1001101), \\ &(1000011), (0101011), (0010111), (1111111)\} . \end{aligned}$$

It is clear that $H \setminus Q_2^7 \cong K_8$. Assume we have a commutative diagram

$$\begin{tikzcd} Q_2^7 \arrow[r, "\psi"] \arrow[d, "p"'] & Q_2^7 \arrow[d, "p"] \\ K_8 \cong H \setminus Q_2^7 \arrow[r, "\bar{\psi}"] & K_8 \cong H \setminus Q_2^7 . \end{tikzcd} \tag{16}$$

Set $\bar{0} = H$ and $\bar{i} = e_i + H$ for $i = 1, 2, \ldots, 7$. Let $\bar{\psi}$ be the automorphism $\bar{\psi} = (\bar{0})(\bar{1}, \bar{2}, \ldots, \bar{7})$. We can assume that $\psi(0) = 0$ (which forces $\bar{\psi}(\bar{0}) = \bar{0}$). Diagram chasing now gives $(1000000) \overset{p}{\mapsto} \bar{1} \overset{\bar{\psi}}{\mapsto} \bar{2}$. Since $\psi(0) = 0$ and $\psi(1000000)$ are adjacent, $\psi(1000000)$ must be the element in $\bar{2}$ adjacent to 0. Thus, we obtain $\psi(1000000) = (0100000)$. In the same way we conclude that $\psi(0100000) = (0001000)$. The element $(1100000) \in \bar{4}$ is adjacent to *i)* (1000000) and *ii)* (0100000). Since $\bar{\psi} \circ p(1100000) = \bar{5}$ we obtain by the fact *i)* that $\psi(1100000)$ is the element in $\bar{5}$ adjacent to $\psi(1000000) = (0100000)$, forcing us to conclude that $\psi(1100000) = (0100001)$. On the other hand, if there is an automorphism ψ, $\psi(1100000)$ must also be the element of $\bar{5}$ adjacent to $\psi(0100000) = (0010000)$, and thus $\psi(1100000) = (1010000)$, which is impossible. The situation is illustrated in figure 11.

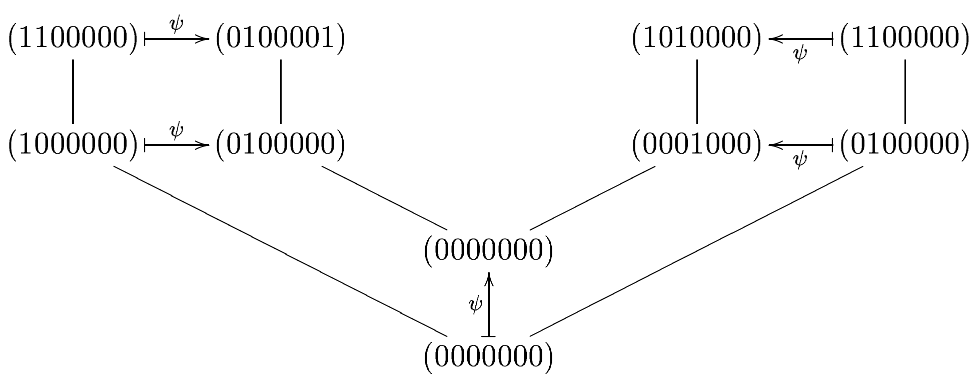

Fig. 11. The impossibility of lifting $\bar{\psi}$: By following two different paths in Q_2^7 from zero to (1100000) two different values for $\psi(1100000)$ are obtained.

Corollary 8 *Let n be a natural number. Then we have* $2^n \equiv 0 \mod n+1$

if and only if there exists a subgroup $G < \mathbb{F}_2^n$ with the property $\mathbb{F}_2^n = G(0) \cup \bigcup_{i=1}^n G(e_i)$.

PROOF. Suppose we have $2^n \equiv 0 \mod n+1$. Since $\mathbb{F}_2^n$ is a p-group there exists a subgroup in its decomposition series $H < \mathbb{F}_2^n$ with the property $[\mathbb{F}_2^n : H] = n+1$. According to Lemma 6, there exists some set of H-representatives $\{\varphi_1, \dots, \varphi_n\}$ that forms a basis of $\mathbb{F}_2^n$. Let f be the $\mathbb{F}_2$-homomorphism defined by $f(\varphi_i) = e_i$, for $i = 1, \dots, n$. Clearly, $G = f(H)$ has the property $\mathbb{F}_2^n = G(0) \cup \bigcup_{i=1}^n G(e_i)$, whence the corollary.

The next Corollary proves the existence of covering maps $\varphi : Q_2^n \longrightarrow K_{n+1}$ provided $2^n \equiv 0 \mod n+1$ holds. This existence was used in the proof of Proposition 5.

Corollary 9 *There exists a covering map*

$$\varphi : Q_2^n \longrightarrow K_{n+1} \tag{17}$$

if and only if $2^n \equiv 0 \mod n+1$ *holds.*

PROOF. Let $\varphi : Q_2^n \longrightarrow K_{n+1}$ be a covering map and let $U = \varphi^{-1}(1)$.
Claim. U is dense in Q_2^n and $\overline{U} = \bigcup_{u \in U} \overline{\{u\}}$.
Let x be an arbitrary Q_2^n vertex. In case of $x \in U$ we are done, otherwise we consider $\varphi(x)$ and 1 in K_{n+1}. Clearly there exists some K_{n+1}-edge of the form $\{\varphi(x), 1\}$ and local bijectivity guarantees that there exists some Q_2^n-edge connecting x and some $u \in U$. Accordingly, for every $x \in \mathrm{v}[Q_2^n]$ $B_1(x) \cap U \neq \varnothing$ holds.
Local bijectivity immediately implies that any two elements $u, u' \in U$ have distance ≥ 3, whence the second assertion follows and the Claim is proved.
In view of the Claim we have $|\overline{U}| = 2^n$ and since $|\overline{\{u\}}| = n+1$, $|U| = 2^n/n+1$ holds i.e. $2^n \cong 0 \mod n+1$.
Suppose next that $2^n \equiv 0 \mod n+1$ holds. Corollary 8 guarantees the existence of a subgroup (!) $G < \mathbb{F}_2^n$ with the property $\mathbb{F}_2^n = G(0) \cup \bigcup_{i=1}^n G(e_i)$. We set $e_{n+1} = 0$ and obtain according to Proposition 7 the covering mapping $\varphi : Q_2^n \longrightarrow G \setminus Q_2^n$ given by

$$\varphi^{-1}(i) = G(e_i)\ . \tag{18}$$

It remains to show that $K_{n+1} \cong G \setminus Q_2^n$ holds. Obviously, $G \setminus Q_2^n$ contains $n+1$ vertices. Let $i \neq j$ be two $G \setminus Q_2^n$ vertices and $x \in G(e_i)$. We observe that each of the sets $G(e_s)$ for $s = 1, \dots, n, n+1$ is is dense in Q_2^n from which we can conclude that there exists some $y \in G(e_j)$ that is adjacent to x. Accordingly, any two non-equal $G \setminus Q_2^n$-vertices are adjacent in $G \setminus Q_2^n$, whence $G \setminus Q_2^n \cong K_{n+1}$.

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